

# Pose-Tracking Controller for Satellites with Time-Varying Inertia

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Satellite proximity operations have been identified by NASA and the USAF as a crucial technology that could enable a series of new missions in space. Such missions would require a satellite to simultaneously and accurately track time-varying relative position and attitude profiles. Moreover, the mass and moment of inertia of a satellite are also typically time-varying, which makes this problem even more challenging. Based on recent results in dual quaternions, a nonlinear adaptive position and attitude tracking controller for satellites with unknown and time-varying mass and inertia matrix is proposed. Dual quaternions are used to represent jointly the position and attitude of the satellite. The controller is shown to ensure almost global asymptotic stability of the combined translational and rotational position and velocity tracking errors. Moreover, sufficient conditions on the reference motion are provided that guarantee mass and inertia matrix identification. The controller compensates for the gravity force, the gravity-gradient torque, Earth's oblateness, and unknown constant disturbance forces and torques. The proposed controller is especially suited for satellites with relatively high and quick variations of mass and moment of inertia, such as highly maneuverable small satellites equipped with relatively powerful thrusters and control moment gyros.

## I. Introduction

The capability to safely and efficiently operate a satellite in close proximity to another small body, such as another satellite or a small asteroid, is a game-changing technology that can potentiate missions such as on-orbit satellite inspection, health monitoring, surveillance, servicing, refueling, and asteroid retrieval. One of the biggest challenges introduced by these space missions is the need to simultaneously and accurately track both time-varying relative position and attitude, i.e., pose, reference trajectories in order to avoid collisions and achieve overall mission objectives. As it will be shown, the problem of simultaneously tracking both position and attitude profiles in space can be substantially simplified through the use of dual quaternions.

Previous controllers developed to address this problem include the adaptive nonlinear controller presented in Ref. [1], which considers unknown – but constant – mass and inertia matrix parameters. This controller has a very high order, which limits its applicability. Ref. [2] also assumes constant but unknown mass and inertia matrix. However, as explained in Ref. [3], if the reference motion is not exciting enough, this controller cannot guarantee convergence of the position and attitude errors to zero. The controller proposed in Ref. [4] also assumes constant mass and inertia matrix, but requires knowledge of an upper bound on the mass and on the maximum eigenvalue of the inertia matrix. It also requires knowledge of upper bounds on several states and parameters.

More recently, an adaptive pose (i.e., position and attitude) tracking controller has been proposed based on dual quaternions<sup>5</sup> that, unlike the controller presented in Ref. [4], does not require a priori knowledge of any upper bounds, ensures *almost*<sup>a</sup> global asymptotic stability of the linear and angular position and velocity tracking errors, and can identify the mass and inertia matrix of the satellite (if the reference motion

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<sup>a</sup>The terminology *almost* global asymptotic stability will be used to designate asymptotic stability over an open and dense set. This is the best one can achieve with a continuous controller for the rotational motion,<sup>6</sup> since the special group of rotation matrices  $SO(3)$  is a compact manifold.

satisfies certain sufficient conditions). This controller has been further improved in Ref. [7] to handle the gravity force, the gravity-gradient torque, constant unknown disturbance forces and torques, and Earth's oblateness (which is typically the largest perturbation on a satellite below GEO<sup>8</sup>). However, Refs. [5] and [7] also assume that the mass and inertia matrix of the satellite are constant.

All the controllers mentioned before assume that the mass and inertia matrix of the satellite are constant (albeit unknown). However, the mass and inertia matrix of a satellite are typically not constant. For example, consider a small satellite equipped with thrusters and control moment gyros. In this case, the mass of the propellant and the moment of inertia of the control moment gyros might be substantial compared to the total mass and moment of inertia of the satellite.

The main contribution of this paper is to remove the assumption of constant mass and inertia matrix used in Refs. [5] and [7].

Dual quaternions are used in this paper to derivate the controller. Dual quaternions are an extension of classical quaternions and provide a compact representation of the position and attitude of a frame with respect to another frame. Their properties, examples of previous applications, and a comparison with other pose representations are discussed in length in Ref. [7]. However, the property that makes them most appealing is that the combined translational and rotational kinematic and dynamic equations of motion written in terms of dual quaternions have the same form as the rotational-only kinematic and dynamic equations of motion written in terms of quaternions. This interesting property has been recently used in Refs. [5,7,9,10] to extend existing attitude-only controllers with some desired properties into position *and* attitude controllers with equivalent desired properties. This was achieved by, essentially, replacing quaternions by dual quaternions in the attitude-only control laws and corresponding Lyapunov functions. This nice property of dual quaternions is used here to extend the attitude-only results presented in Refs. [11,12].

## II. Mathematical Preliminaries

For the benefit of the reader, the main properties of quaternions and dual quaternions are summarized here. Additional information can be found in Ref. [7].

### A. Quaternions

A quaternion can be represented as  $q = q_1i + q_2j + q_3k + q_4$ , where  $q_1, q_2, q_3, q_4 \in \mathbb{R}$  and  $i, j$ , and  $k$  satisfy  $i^2 = j^2 = k^2 = -1$ ,  $i = jk = -kj$ ,  $j = ki = -ik$ , and  $k = ij = -ji$ ,<sup>13</sup> or alternatively as an ordered pair  $q = (\bar{q}, q_4)$ , where  $\bar{q} = [q_1 \ q_2 \ q_3]^T \in \mathbb{R}^3$  is the *vector part* and  $q_4 \in \mathbb{R}$  is the *scalar part*. Quaternions with zero vector part and with zero scalar part are called *scalar quaternions* and *vector quaternions*, respectively. The set of quaternions will be denoted by  $\mathbb{H} = \{q : q = q_1i + q_2j + q_3k + q_4, q_1, q_2, q_3, q_4 \in \mathbb{R}\}$ , the set of scalar quaternions will be denoted by  $\mathbb{H}^s = \{q \in \mathbb{H} : q_1 = q_2 = q_3 = 0\}$ , and the set of vector quaternions will be denoted by  $\mathbb{H}^v = \{q \in \mathbb{H} : q_4 = 0\}$ . Moreover, the quaternions  $(0, 1)$  and  $(0, 0)$  will be denoted by  $1$  and  $0$ , respectively. For the definition of the basic operations on quaternions and their main properties, the reader is referred to Ref. [7].

### B. Dual Quaternions

A dual quaternion is defined as  $\hat{q} = q_r + \epsilon q_d$ , where  $\epsilon$  is the *dual unit* that satisfies  $\epsilon^2 = 0$  and  $\epsilon \neq 0$ . The quaternion  $q_r \in \mathbb{H}$  is called the *real part* of the dual quaternion, whereas the quaternion  $q_d \in \mathbb{H}$  is called the *dual part* of the dual quaternion.

Dual quaternions formed from vector quaternions (i.e.,  $q_r, q_d \in \mathbb{H}^v$ ) and from scalar quaternions (i.e.,  $q_r, q_d \in \mathbb{H}^s$ ) are called *dual vector quaternions* and *dual scalar quaternions*, respectively. The set of dual quaternions will be denoted by  $\mathbb{H}_d = \{\hat{q} : \hat{q} = q_r + \epsilon q_d, q_r, q_d \in \mathbb{H}\}$ , the set of dual scalar quaternions will be denoted by  $\mathbb{H}_d^s = \{\hat{q} : \hat{q} = q_r + \epsilon q_d, q_r, q_d \in \mathbb{H}^s\}$ , the set of dual vector quaternions will be denoted by  $\mathbb{H}_d^v = \{\hat{q} : \hat{q} = q_r + \epsilon q_d, q_r, q_d \in \mathbb{H}^v\}$ , and the set of dual scalar quaternions with zero dual part will be denoted by  $\mathbb{H}_d^r = \{\hat{q} : \hat{q} = q_r + \epsilon 0, q_r \in \mathbb{H}^s\}$ . Moreover, the dual quaternions  $1 + \epsilon 0$  and  $0 + \epsilon 0$  will be denoted by  $\hat{1}$  and  $\hat{0}$ , respectively. Again, for the definition of the basic operations on dual quaternions and their main properties, the reader is referred to Ref. [7].

### 1. Representing Attitude and Position with Unit Dual Quaternions.

The position and attitude (i.e., pose) of a body-fixed frame with respect to an inertial frame can be represented by a *unit* quaternion  $q_{B/I} \in \mathbb{H}^u = \{q \in \mathbb{H} : q \cdot q = 1\}$  and by a translation vector, respectively. Alternatively, the pose of the body frame with respect to the inertial frame can be represented more compactly by the *unit* dual quaternion<sup>14</sup>  $\hat{q}_{B/I} = q_{B/I} + \epsilon \frac{1}{2} r_{B/I}^I q_{B/I} = q_{B/I} + \epsilon \frac{1}{2} q_{B/I} r_{B/I}^B$ , where  $r_{Y/Z}^x = (\bar{r}_{Y/Z}^x, 0)$  and  $\bar{r}_{Y/Z}^x = [x_{Y/Z}^x \ y_{Y/Z}^x \ z_{Y/Z}^x]^T$  is the translation vector from the origin of the Z-frame to the origin of the Y-frame expressed in the X-frame. A unit dual quaternion is defined as belonging to the set<sup>9</sup>  $\mathbb{H}_d^u = \{\hat{q} \in \mathbb{H}_d : \hat{q} \cdot \hat{q} = \hat{q} \hat{q}^* = \hat{q}^* \hat{q} = \|\hat{q}\|_d = \hat{1}\}$ .

### 2. Representing the Rotational and Translational Kinematic Equations with Dual Quaternions

The kinematic equations of a body-fixed frame and of a frame with some desired pose, both with respect to an inertial frame and represented by the unit dual quaternions  $\hat{q}_{B/I}$  and  $\hat{q}_{D/I} = q_{D/I} + \epsilon \frac{1}{2} r_{D/I}^I q_{D/I} = q_{D/I} + \epsilon \frac{1}{2} q_{D/I} r_{D/I}^D$ , respectively, are given by<sup>14</sup>  $\dot{\hat{q}}_{B/I} = \frac{1}{2} \hat{\omega}_{B/I}^I \hat{q}_{B/I} = \frac{1}{2} \hat{q}_{B/I} \hat{\omega}_{B/I}^B$  and  $\dot{\hat{q}}_{D/I} = \frac{1}{2} \hat{\omega}_{D/I}^I \hat{q}_{D/I} = \frac{1}{2} \hat{q}_{D/I} \hat{\omega}_{D/I}^D$ , where  $\hat{\omega}_{Y/Z}^x$  is the *dual velocity* of the Y-frame with respect to the Z-frame expressed in the X-frame, so that  $\hat{\omega}_{Y/Z}^x = \omega_{Y/Z}^x + \epsilon(v_{Y/Z}^x + \omega_{Y/Z}^x \times r_{X/Y}^x)$ ,  $v_{Y/Z}^x = (\bar{v}_{Y/Z}^x, 0)$ ,  $\bar{v}_{Y/Z}^x$  is the linear velocity of the Y-frame with respect to the Z-frame expressed in the X-frame,  $\omega_{Y/Z}^x = (\bar{\omega}_{Y/Z}^x, 0)$ , and  $\bar{\omega}_{Y/Z}^x$  is the angular velocity of the Y-frame with respect to the Z-frame expressed in the X-frame.

The relative pose between the body frame and the desired frame is represented by the *unit*<sup>9</sup> dual quaternion<sup>4,10,13,15</sup>  $\hat{q}_{B/D} = \hat{q}_{D/I}^* \hat{q}_{B/I} = q_{B/D} + \epsilon \frac{1}{2} q_{B/D} r_{B/D}^B$ , where  $r_{B/D}^B = r_{B/I}^B - r_{D/I}^B$ . The kinematic equations of  $\hat{q}_{B/D}$  can be calculated to be<sup>4,10,15</sup>

$$\dot{\hat{q}}_{B/D} = \frac{1}{2} \hat{q}_{B/D} \hat{\omega}_{B/D}^B = \frac{1}{2} \hat{\omega}_{B/D}^D \hat{q}_{B/D}, \quad (1)$$

where  $\hat{\omega}_{B/D}^B = \hat{\omega}_{B/I}^B - \hat{\omega}_{D/I}^B$  is the relative dual velocity between the body frame and the desired frame expressed in the body frame. Note that  $\hat{\omega}_{D/I}^B = \hat{q}_{B/D}^* \hat{\omega}_{D/I}^D \hat{q}_{B/D}$  and  $\hat{\omega}_{B/I}^D = \hat{q}_{B/D} \hat{\omega}_{B/I}^B \hat{q}_{B/D}^*$ .

## III. Dual Quaternion Representation of the Relative Dynamic Equations for a Satellite with Time-Varying Mass and Inertia Matrix

The dynamic equations of motion of a rigid body with constant mass and inertia matrix represented in terms of dual quaternions can be found in Ref. [10]. However, as mentioned before, the mass and inertia matrix of a spacecraft are not constant in general. In this paper, this variation will be modeled by rewriting the previous dynamic equations of motion as follows:

$$(\hat{\omega}_{B/D}^B)^s = (M^B(t))^{-1} \star \left( \hat{f}^B - (\hat{\omega}_{B/D}^B + \hat{\omega}_{D/I}^B) \times (M^B(t) \star ((\hat{\omega}_{B/D}^B)^s + (\hat{\omega}_{D/I}^B)^s)) \right) - (\hat{q}_{B/D}^* \hat{\omega}_{D/I}^D \hat{q}_{B/D})^s - (\hat{\omega}_{D/I}^B \times \hat{\omega}_{B/D}^B)^s, \quad (2)$$

where  $\hat{f}^B = f^B + \epsilon \tau^B$  is the total *dual force* applied to the body about its center of mass expressed in the body frame,  $f^B = (\bar{f}^B, 0)$ ,  $\bar{f}^B$  is the total force vector applied to the body,  $\tau^B = (\bar{\tau}^B, 0)$ , and  $\bar{\tau}^B$  is the total moment vector applied to the body about its center of mass,  $M^B(t) \in \mathbb{R}^{8 \times 8}$  is the *dual inertia matrix*<sup>9</sup> defined as

$$M^B(t) = \begin{bmatrix} m(t)I_3 & 0_{3 \times 1} & 0_{3 \times 3} & 0_{3 \times 1} \\ 0_{1 \times 3} & 1 & 0_{1 \times 3} & 0 \\ 0_{3 \times 3} & 0_{3 \times 1} & \bar{I}^B(t) & 0_{3 \times 1} \\ 0_{1 \times 3} & 0 & 0_{1 \times 3} & 1 \end{bmatrix}, \quad \bar{I}^B(t) = \begin{bmatrix} I_{11}(t) & I_{12}(t) & I_{13}(t) \\ I_{12}(t) & I_{22}(t) & I_{23}(t) \\ I_{13}(t) & I_{23}(t) & I_{33}(t) \end{bmatrix}, \quad I^B(t) = \begin{bmatrix} \bar{I}^B(t) & 0_{3 \times 1} \\ 0_{1 \times 3} & 1 \end{bmatrix}, \quad (3)$$

$\bar{I}^B(t) \in \mathbb{R}^{3 \times 3}$  is the mass moment of inertia of the body about its center of mass expressed in the body frame, and  $m(t)$  is the mass of the body. Note that the dual inertia matrix is symmetric and positive definite.

Although the time-varying dual inertia matrix in Eq. (2) is assumed to be unknown,  $M^B(t)$  is assumed to be known. That is typically the case in a spacecraft equipped with thrusters and control moment gyros. Whereas the time-varying mass and inertia matrix of the spacecraft might be unknown, their time derivatives are typically known, as  $\dot{m}(t)$  and  $\dot{I}^B(t)$  are directly related with the control forces and moments produced by the spacecraft.

Like in Ref. [7], the total dual force acting on the body will be decomposed as follows:

$$\hat{f}^B = \hat{f}_g^B + \hat{f}_{Vg}^B + \hat{f}_{J_2}^B + \hat{f}_d^B + \hat{f}_c^B, \quad (4)$$

where  $\hat{f}_g^B = m(t)\hat{a}_g^B$ ,  $\hat{a}_g^B = a_g^B + \epsilon 0$ ,  $a_g^B = (\bar{a}_g^B, 0)$ ,  $\bar{a}_g^B$  is the gravitational acceleration given by  $\bar{a}_g^B = -\mu \frac{\hat{r}_{B/I}^B}{\|\hat{r}_{B/I}^B\|^3}$ ,  $\mu = 398600.4418 \text{ km}^3/\text{s}^2$  is Earth's gravitational parameter,<sup>8</sup>  $\hat{f}_{\nabla g}^B = 0 + \epsilon \tau_{\nabla g}^B$ ,  $\tau_{\nabla g}^B = (\bar{\tau}_{\nabla g}^B, 0)$ ,  $\bar{\tau}_{\nabla g}^B$  is the gravity gradient torque<sup>4</sup> given by  $\bar{\tau}_{\nabla g}^B = 3\mu \frac{\hat{r}_{B/I}^B \times (\hat{I}^B(t)\hat{r}_{B/I}^B)}{\|\hat{r}_{B/I}^B\|^5}$ ,  $\hat{f}_{J_2}^B = m(t)\hat{a}_{J_2}^B$ ,  $\hat{a}_{J_2}^B = a_{J_2}^B + \epsilon 0$ ,  $a_{J_2}^B = (\bar{a}_{J_2}^B, 0)$ ,  $\bar{a}_{J_2}^B$  is the perturbing acceleration due to Earth's oblateness<sup>16</sup> given by

$$\bar{a}_{J_2}^B = -\frac{3}{2} \frac{\mu J_2 R_e^2}{\|\hat{r}_{B/I}^B\|^4} \begin{bmatrix} (1 - 5(\frac{z_{B/I}^B}{\|\hat{r}_{B/I}^B\|})^2) \frac{x_{B/I}^B}{\|\hat{r}_{B/I}^B\|} \\ (1 - 5(\frac{z_{B/I}^B}{\|\hat{r}_{B/I}^B\|})^2) \frac{y_{B/I}^B}{\|\hat{r}_{B/I}^B\|} \\ (3 - 5(\frac{z_{B/I}^B}{\|\hat{r}_{B/I}^B\|})^2) \frac{z_{B/I}^B}{\|\hat{r}_{B/I}^B\|} \end{bmatrix}, \quad (5)$$

$J_2 = 0.0010826267$ ,  $R_e = 6378.137 \text{ km}$  is Earth's mean equatorial radius,<sup>8</sup>  $\hat{f}_c^B = f_c^B + \epsilon \tau_c^B$  is the dual control force, and  $\hat{f}_d^B = f_d^B + \epsilon \tau_d^B$  is the dual disturbance force. Like in Ref. [7],  $\hat{f}_d^B$  is assumed to be an unknown constant that captures neglected disturbance forces and torques, such as atmospheric drag, solar radiation, and third-bodies. Finally, note that  $\hat{f}_g^B$ ,  $\hat{f}_{\nabla g}^B$ , and  $\hat{f}_{J_2}^B$  can be written more compactly in terms of dual quaternions as  $\hat{f}_g^B = M^B(t) \star \hat{a}_g^B$ ,  $\hat{f}_{\nabla g}^B = \frac{3\mu \hat{r}_{B/I}^B}{\|\hat{r}_{B/I}^B\|^5} \times (M^B(t) \star (\hat{r}_{B/I}^B)^s)$ , and  $\hat{f}_{J_2}^B = M^B(t) \star \hat{a}_{J_2}^B$ , where  $\hat{r}_{B/I}^B = r_{B/I}^B + \epsilon 0$ .

In a spacecraft equipped with thrusters and control moment gyros, the dual control force is a known function of the time derivative of the mass and inertia matrix of the spacecraft. In other words, to produce control forces and torques, thrusters and control moment gyros need to change the mass and inertia matrix of the spacecraft, respectively. In this paper, this effect will be modeled by considering the dual control force to be a known function of the time derivative of the dual inertia matrix, i.e.,  $\hat{f}_c^B = \hat{f}_c^B(\dot{M}^B(t))$ . The exact form of this function depends on the characteristics of the actuators. This issue is further discussed in Section IV.

#### IV. Pose-Tracking Controller for Spacecraft with Unknown Time-Varying Mass and Inertia Matrix

The main result of this paper is an adaptive pose-tracking controller for spacecraft with unknown time-varying mass and inertia matrix. This controller is presented in the next theorem and shown to ensure *almost* global asymptotic stability of the linear and angular position and velocity tracking errors. To simplify the notation, the time dependence of  $M^B(t)$  and  $\dot{M}^B(t)$  will not be explicitly indicated from now on.

**Theorem 1.** *Let the dual control force in Eq. (4) be defined by the feedback control law*

$$\begin{aligned} \hat{f}_c^B = & -\widehat{M}^B \star \hat{a}_g^B - \frac{3\mu \hat{r}_{B/I}^B}{\|\hat{r}_{B/I}^B\|^5} \times (\widehat{M}^B \star (\hat{r}_{B/I}^B)^s) - \widehat{M}^B \star \hat{a}_{J_2}^B - \widehat{f}_d^B - \text{vec}(\hat{q}_{B/D}^* (\hat{q}_{B/D}^s - \hat{1}^s)) - K_d \star \hat{s}^s - \frac{1}{2} \dot{M}^B \star \hat{s}^s \\ & + \hat{\omega}_{B/I}^B \times (\widehat{M}^B \star (\hat{\omega}_{B/I}^B)^s) + \widehat{M}^B \star (\hat{q}_{B/D}^* \hat{\omega}_{D/I}^D \hat{q}_{B/D}^s) + \widehat{M}^B \star (\hat{\omega}_{D/I}^B \times \hat{\omega}_{B/D}^B)^s - \widehat{M}^B \star (K_p \star \frac{d}{dt}(\hat{q}_{B/D}^* (\hat{q}_{B/D}^s - \hat{1}^s))), \end{aligned} \quad (6)$$

where

$$\hat{s} = \hat{\omega}_{B/D}^B + (K_p \star (\hat{q}_{B/D}^* (\hat{q}_{B/D}^s - \hat{1}^s)))^s, \quad K_p = \begin{bmatrix} K_r & 0_{4 \times 4} \\ 0_{4 \times 4} & K_q \end{bmatrix}, \quad K_d = \begin{bmatrix} K_v & 0_{4 \times 4} \\ 0_{4 \times 4} & K_\omega \end{bmatrix}, \quad (7)$$

$$K_r = \begin{bmatrix} \bar{K}_r & 0_{3 \times 1} \\ 0_{1 \times 3} & 0 \end{bmatrix}, K_q = \begin{bmatrix} \bar{K}_q & 0_{3 \times 1} \\ 0_{1 \times 3} & 0 \end{bmatrix}, K_v = \begin{bmatrix} \bar{K}_v & 0_{3 \times 1} \\ 0_{1 \times 3} & 0 \end{bmatrix}, K_\omega = \begin{bmatrix} \bar{K}_\omega & 0_{3 \times 1} \\ 0_{1 \times 3} & 0 \end{bmatrix}, \quad (8)$$

$\bar{K}_r, \bar{K}_q, \bar{K}_v, \bar{K}_\omega \in \mathbb{R}^{3 \times 3}$  are positive definite matrices,  $\widehat{M}^B$  is an estimate of the dual inertia matrix updated according to  $\frac{d}{dt} \text{v}(\widehat{M}^B) = \text{v}(\dot{M}^B) + \frac{d}{dt} \text{v}(\Delta M^B)$ ,

$$\begin{aligned} \frac{d}{dt} \text{v}(\Delta M^B) = & K_i [h(\hat{s}^s, -(\hat{q}_{B/D}^* \hat{\omega}_{D/I}^D \hat{q}_{B/D}^s) - (\hat{\omega}_{D/I}^B \times \hat{\omega}_{B/D}^B)^s) + K_p \star \frac{d(\hat{q}_{B/D}^* (\hat{q}_{B/D}^s - \hat{1}^s))}{dt} + \hat{a}_g^B + \hat{a}_{J_2}^B) \\ & - h((\hat{s} \times \hat{\omega}_{B/I}^B)^s, (\hat{\omega}_{B/I}^B)^s) + h((\hat{s} \times \frac{3\mu \hat{r}_{B/I}^B}{\|\hat{r}_{B/I}^B\|^5})^s, (\hat{r}_{B/I}^B)^s)], \end{aligned} \quad (9)$$

$K_i \in \mathbb{R}^{7 \times 7}$  is a positive definite matrix,  $v(M^B) = [I_{11} \ I_{12} \ I_{13} \ I_{22} \ I_{23} \ I_{33} \ m]^T$  is a vectorized version of the dual inertia matrix  $M^B$ , the function  $h: \mathbb{H}_d^u \times \mathbb{H}_d^u \rightarrow \mathbb{R}^7$  is defined as  $\hat{a} \circ (M^B \star \hat{b}) = h(\hat{a}, \hat{b})^T v(M^B) = v(M^B)^T h(\hat{a}, \hat{b})$  or, equivalently,  $h(\hat{a}, \hat{b}) = [a_5 b_5, a_6 b_5 + a_5 b_6, a_7 b_5 + a_5 b_7, a_6 b_6, a_7 b_6 + a_6 b_7, a_7 b_7, a_1 b_1 + a_2 b_2 + a_3 b_3]^T$ ,  $\hat{f}_d^B$  is an estimate of the dual disturbance force updated according to

$$\frac{d}{dt} \hat{f}_d^B = \frac{d}{dt} \Delta \hat{f}_d^B = K_j \star \hat{s}^s, \quad K_j = \begin{bmatrix} K_f & 0_{4 \times 4} \\ 0_{4 \times 4} & K_\tau \end{bmatrix}, \quad K_f = \begin{bmatrix} \bar{K}_f & 0_{3 \times 1} \\ 0_{1 \times 3} & 1 \end{bmatrix}, \quad K_\tau = \begin{bmatrix} \bar{K}_\tau & 0_{3 \times 1} \\ 0_{1 \times 3} & 1 \end{bmatrix}, \quad (10)$$

and  $\bar{K}_f, \bar{K}_\tau \in \mathbb{R}^{3 \times 3}$  are positive definite matrices. Let  $\dot{M}^B$  be known and assume that  $\hat{q}_{B/D}, \hat{\omega}_{D/I}^B, \dot{\hat{\omega}}_{D/I}^B, v(M^B), v(\dot{M}^B) \in \mathcal{L}_\infty$  and  $\hat{r}_{B/I}^B \neq \hat{0}$ . Then, for all initial conditions,  $\lim_{t \rightarrow \infty} \hat{q}_{B/D} = \pm \hat{1}$  (i.e.,  $\lim_{t \rightarrow \infty} q_{B/D} = \pm 1$  and  $\lim_{t \rightarrow \infty} r_{B/D}^B = 0$ ),  $\lim_{t \rightarrow \infty} \hat{\omega}_{B/D}^B = \hat{0}$  (i.e.,  $\lim_{t \rightarrow \infty} \omega_{B/D}^B = 0$  and  $\lim_{t \rightarrow \infty} v_{B/D}^B = 0$ ), and  $v(\widehat{M}^B), \hat{f}_d^B \in \mathcal{L}_\infty$ .

*Proof.* Let the dual inertia matrix and dual disturbance force estimation errors be defined as

$$\Delta M^B = \widehat{M}^B - M^B \quad \text{and} \quad \Delta \hat{f}_d^B = \widehat{\hat{f}}_d^B - \hat{f}_d^B, \quad (11)$$

respectively. Note that  $(\hat{q}_{B/D}, \hat{s}, v(\Delta M^B), \Delta \hat{f}_d^B) = (\pm \hat{1}, \hat{0}, 0_{7 \times 1}, \hat{0})$  are the equilibrium conditions of the closed-loop system formed by Eqs. (2), (4), (1), (9), and (10). Consider now the following time-varying candidate Lyapunov function for the equilibrium point  $(\hat{q}_{B/D}, \hat{s}, v(\Delta M^B), \Delta \hat{f}_d^B) = (+\hat{1}, \hat{0}, 0_{7 \times 1}, \hat{0})$ :

$$V(\hat{q}_{B/D}, \hat{s}, v(\Delta M^B), \Delta \hat{f}_d^B) = (\hat{q}_{B/D} - \hat{1}) \circ (\hat{q}_{B/D} - \hat{1}) + \frac{1}{2} \hat{s}^s \circ (M^B \star \hat{s}^s) + \frac{1}{2} v(\Delta M^B)^T K_i^{-1} v(\Delta M^B) + \frac{1}{2} \Delta \hat{f}_d^B \circ (K_j^{-1} \star \Delta \hat{f}_d^B). \quad (12)$$

Note that  $V(\hat{q}_{B/D} = \hat{1}, \hat{s} = \hat{0}, v(\Delta M^B) = 0_{7 \times 1}, \Delta \hat{f}_d^B = \hat{0}) = 0$  and  $\min\{1, \frac{1}{2} \sigma_{\min}(M^B), \frac{1}{2} \frac{1}{\sigma_{\max}(K_i)}, \frac{1}{2} \frac{1}{\sigma_{\max}(K_j)}\} (\|\hat{q}_{B/D} - \hat{1}\|^2 + \|\hat{s}\|^2 + \|v(\Delta M^B)\|^2 + \|\Delta \hat{f}_d^B\|^2) \leq V(\hat{q}_{B/D}, \hat{s}, v(\Delta M^B), \Delta \hat{f}_d^B) \leq \max\{1, \frac{1}{2} \sigma_{\max}(M^B), \frac{1}{2} \frac{1}{\sigma_{\min}(K_i)}, \frac{1}{2} \frac{1}{\sigma_{\min}(K_j)}\} (\|\hat{q}_{B/D} - \hat{1}\|^2 + \|\hat{s}\|^2 + \|v(\Delta M^B)\|^2 + \|\Delta \hat{f}_d^B\|^2)$  for all  $(\hat{q}_{B/D}, \hat{s}, v(\Delta M^B), \Delta \hat{f}_d^B) \in \mathbb{H}_d^u \times \mathbb{H}_d^u \times \mathbb{R}^7 \times \mathbb{H}_d^u \setminus \{\hat{1}, \hat{0}, 0_{7 \times 1}, \hat{0}\}$ , where  $\sigma_{\min}$  and  $\sigma_{\max}$  denote the minimum and maximum singular value, respectively. In addition, the time derivative of  $V$  is equal to

$$\dot{V} = 2(\hat{q}_{B/D} - \hat{1}) \circ \dot{\hat{q}}_{B/D} + \hat{s}^s \circ (M^B \star \dot{\hat{s}}^s) + \frac{1}{2} \hat{s}^s \circ (\dot{M}^B \star \hat{s}^s) + v(\Delta M^B)^T K_i^{-1} \frac{d}{dt} v(\Delta M^B) + \Delta \hat{f}_d^B \circ (K_j^{-1} \star \frac{d}{dt} \Delta \hat{f}_d^B).$$

Then, since from Eq. (1),  $\hat{\omega}_{B/D}^B = 2\hat{q}_{B/D}^* \dot{\hat{q}}_{B/D}$ , Eq. (7) can be rewritten as  $\dot{\hat{q}}_{B/D} = \frac{1}{2} \hat{q}_{B/D} \hat{s} - \frac{1}{2} \hat{q}_{B/D} (K_p \star (\hat{q}_{B/D}^* (\hat{q}_{B/D}^s - \hat{1}^s)))^s$ , which can then be plugged into  $\dot{V}$ , together with the time derivative of Eq. (7), to yield

$$\begin{aligned} \dot{V} = & (\hat{q}_{B/D} - \hat{1}) \circ (\hat{q}_{B/D} \hat{s} - \hat{q}_{B/D} (K_p \star (\hat{q}_{B/D}^* (\hat{q}_{B/D}^s - \hat{1}^s)))^s) + \hat{s}^s \circ (M^B \star (\dot{\hat{\omega}}_{B/D}^B)^s) + \frac{1}{2} \hat{s}^s \circ (\dot{M}^B \star \hat{s}^s) \\ & + \hat{s}^s \circ (M^B \star (K_p \star \frac{d(\hat{q}_{B/D}^* (\hat{q}_{B/D}^s - \hat{1}^s))}{dt})) + v(\Delta M^B)^T K_i^{-1} \frac{d}{dt} v(\Delta M^B) + \Delta \hat{f}_d^B \circ (K_j^{-1} \star \frac{d}{dt} \Delta \hat{f}_d^B). \end{aligned} \quad (13)$$

Applying Eq. (9) of Ref. [5], inserting Eq. (2), and using  $\hat{\omega}_{B/D}^B + \hat{\omega}_{D/I}^B = \hat{\omega}_{B/I}^B$  yields

$$\begin{aligned} \dot{V} = & \hat{s}^s \circ (\hat{q}_{B/D}^* (\hat{q}_{B/D}^s - \hat{1}^s)) - (K_p \star (\hat{q}_{B/D}^* (\hat{q}_{B/D}^s - \hat{1}^s))) \circ (\hat{q}_{B/D}^* (\hat{q}_{B/D}^s - \hat{1}^s)) + \hat{s}^s \circ (\hat{f}^B - \hat{\omega}_{B/I}^B \times (M^B \star (\hat{\omega}_{B/I}^B)^s) \\ & - M^B \star (\hat{q}_{B/D}^* \dot{\hat{\omega}}_{D/I}^B \hat{q}_{B/D}^s) - M^B \star (\hat{\omega}_{D/I}^B \times \hat{\omega}_{B/D}^B)^s + \frac{1}{2} \hat{s}^s \circ (\dot{M}^B \star \hat{s}^s) + \hat{s}^s \circ (M^B \star (K_p \star \frac{d(\hat{q}_{B/D}^* (\hat{q}_{B/D}^s - \hat{1}^s))}{dt})) \\ & + v(\Delta M^B)^T K_i^{-1} \frac{d}{dt} v(\Delta M^B) + \Delta \hat{f}_d^B \circ (K_j^{-1} \star \frac{d}{dt} \Delta \hat{f}_d^B). \end{aligned} \quad (14)$$

Introducing the feedback control law given by Eq. (6), using Eq. (10) and Eq. (13) of Ref. [5], and the commutativity property of the dual quaternion circle product yields

$$\begin{aligned} \dot{V} = & -(\hat{q}_{B/D}^* (\hat{q}_{B/D}^s - \hat{1}^s)) \circ (K_p \star (\hat{q}_{B/D}^* (\hat{q}_{B/D}^s - \hat{1}^s))) + (\hat{s} \times \hat{\omega}_{B/I}^B)^s \circ (\Delta M^B \star (\hat{\omega}_{B/I}^B)^s) + \hat{s}^s \circ (\Delta M^B \star (\hat{q}_{B/D}^* \dot{\hat{\omega}}_{D/I}^B \hat{q}_{B/D}^s) \\ & + \Delta M^B \star (\hat{\omega}_{D/I}^B \times \hat{\omega}_{B/D}^B)^s - \Delta M^B \star (K_p \star \frac{d(\hat{q}_{B/D}^* (\hat{q}_{B/D}^s - \hat{1}^s))}{dt}) - \Delta M^B \star \hat{a}_g^B - \Delta M^B \star \hat{a}_{J_2}^B - \Delta \hat{f}_d^B) \\ & - (\hat{s} \times \frac{3\mu \hat{r}_{B/I}^B}{\|\hat{r}_{B/I}^B\|^5})^s \circ (\Delta M^B \star (\hat{r}_{B/I}^B)^s) - \hat{s}^s \circ (K_d \star \hat{s}^s) + v(\Delta M^B)^T K_i^{-1} \frac{d}{dt} v(\Delta M^B) + \Delta \hat{f}_d^B \circ (K_j^{-1} \star \frac{d}{dt} \Delta \hat{f}_d^B). \end{aligned} \quad (15)$$

Therefore, if  $\frac{d}{dt}v(\Delta M^B)$  and  $\frac{d}{dt}\Delta\hat{f}_d^B$  are defined as in Eqs. (9) and (10), it follows that  $\dot{V} = -(\hat{q}_{B/D}^*(\hat{q}_{B/D}^s - \hat{1}^s)) \circ (K_p \star (\hat{q}_{B/D}^*(\hat{q}_{B/D}^s - \hat{1}^s))) - \hat{s}^s \circ (K_d \star \hat{s}^s) \leq 0$ , for all  $(\hat{q}_{B/D}, \hat{s}, v(\Delta M^B), \Delta\hat{f}_d^B) \in \mathbb{H}_d^u \times \mathbb{H}_d^u \times \mathbb{R}^7 \times \mathbb{H}_d^u \setminus \{\hat{1}, \hat{0}, 0_{7 \times 1}, \hat{0}\}$ . Hence, the equilibrium point  $(\hat{q}_{B/D}, \hat{s}, v(\Delta M^B), \Delta\hat{f}_d^B) = (+\hat{1}, \hat{0}, 0_{7 \times 1}, \hat{0})$  is uniformly Lyapunov stable,<sup>17</sup> i.e.,  $\hat{q}_{B/D}, \hat{s}, v(\Delta M^B), \Delta\hat{f}_d^B \in \mathcal{L}_\infty$ . Moreover, from Eqs. (7) and (11), this implies that  $\hat{\omega}_{B/D}^B, v(\widehat{M}^B), \hat{f}_d^B \in \mathcal{L}_\infty$ . In addition, since  $V \geq 0$  and  $\dot{V} \leq 0$ ,  $\lim_{t \rightarrow \infty} V(t)$  exists and is finite. By extension,  $\lim_{t \rightarrow \infty} \int_0^t \dot{V}(\tau) d\tau = \lim_{t \rightarrow \infty} V(t) - V(0)$  also exists and is finite. Since  $\hat{q}_{B/D}, \hat{s}, v(\Delta M^B), \Delta\hat{f}_d^B, \hat{\omega}_{B/D}^B, v(\widehat{M}^B), \hat{f}_d^B, \hat{\omega}_{D/I}^D, \hat{\omega}_{B/I}^B, \hat{q}_{D/I}, v(M^B), v(\dot{M}^B) \in \mathcal{L}_\infty$  and  $\hat{r}_{B/I}^B \neq \hat{0}$ , then from Eqs. (1), (6), and (2) and from Lemma 1 of Ref. [7],  $\hat{r}_{B/I}^B, \hat{q}_{B/D}, \hat{f}^B, \hat{\omega}_{B/D}^B, \hat{s} \in \mathcal{L}_\infty$ . Hence, by Barbalat's lemma,  $\lim_{t \rightarrow \infty} \text{vec}(\hat{q}_{B/D}^*(\hat{q}_{B/D}^s - \hat{1}^s)) = \hat{0}$  and  $\lim_{t \rightarrow \infty} \hat{s} = \hat{0}$ . In Ref. 9, it is shown that  $\lim_{t \rightarrow \infty} \text{vec}(\hat{q}_{B/D}^*(\hat{q}_{B/D}^s - \hat{1}^s)) = \hat{0}$  is equivalent to  $\lim_{t \rightarrow \infty} \hat{q}_{B/D} = \pm\hat{1}$ . Finally,  $\hat{\omega}_{B/D}^B \rightarrow \hat{0}$  results from calculating the limit as  $t \rightarrow \infty$  of both sides of Eq. (7).  $\square$

**Remark 1.** According to Theorem 1,  $\hat{q}_{B/D}$  converges to either  $+\hat{1}$  or  $-\hat{1}$ . Since  $\hat{q}_{B/D} = +\hat{1}$  and  $\hat{q}_{B/D} = -\hat{1}$  represent the same physical pose between frames, either equilibrium is acceptable. However, this can lead to the so-called *unwinding phenomenon*, where a large rotation to the equilibrium (greater than 180 degrees) is performed, despite the fact that a smaller rotation (less than 180 degrees) exists. For more information about this well know problem of quaternions and possible solutions, the reader is referred to Refs. [4, 6, 18, 19].

**Remark 2.** The terms  $\widehat{M}^B \star \hat{a}_g^B, \frac{3\mu\hat{r}_{B/I}^B}{\|\hat{r}_{B/I}^B\|^5} \times (\widehat{M}^B \star (\hat{r}_{B/I}^B)^s), \widehat{M}^B \star \hat{a}_{J_2}^B$ , and  $\hat{f}_d^B$  of the control law given by Eq. (6) are estimates of the gravitational force, gravity-gradient torque, perturbing force due to Earth's oblateness, and dual disturbance force calculated using the estimated mass and inertia matrix. These terms can be thought of as an approximate cancellation of these forces and torques. As shown in Ref. [9], the term  $\text{vec}(\hat{q}_{B/D}^*(\hat{q}_{B/D}^s - \hat{1}^s))$  is equal to  $\frac{1}{2}r_{B/D}^B + \epsilon \text{vec}(q_{B/D})$  and, hence, is the feedback of the relative position and attitude errors. The term  $K_d \star \hat{s}^s$  can be thought of as a damping term, where  $\hat{s}$  takes the place of  $\hat{\omega}_{B/D}^B$ . The terms  $\hat{\omega}_{B/I}^B \times (\widehat{M}^B \star (\hat{\omega}_{B/I}^B)^s), \widehat{M}^B \star (\hat{q}_{B/D}^* \hat{\omega}_{D/I}^D \hat{q}_{B/D})^s$ , and  $\widehat{M}^B \star (\hat{\omega}_{D/I}^B \times \hat{\omega}_{B/D}^B)^s$  are a direct cancellation of identical terms in Eq. (2) with the true mass and inertia matrix replaced by their estimates. The term  $\widehat{M}^B \star (K_p \star \frac{d}{dt}(\hat{q}_{B/D}^*(\hat{q}_{B/D}^s - \hat{1}^s)))$  is a result of using  $\hat{s}$  instead of  $\hat{\omega}_{B/D}^B$  in the damping term and, ultimately, guarantees that the pose error will converge to zero even if the reference motion is not sufficiently exciting. Finally, the term  $\frac{1}{2}\dot{M}^B \star \hat{s}^s$  is a result of having time-varying mass and inertia matrix.

**Remark 3.** As mentioned before, the dual control force is typically a known function of the time derivative of the dual inertia matrix, i.e.,  $\hat{f}_c^B = \hat{f}_c^B(\dot{M}^B)$ . In other words, a spacecraft typically produces control forces and moments by changing its mass and inertia matrix, for example, by using thrusters and control moment gyros, respectively. In that case, the required  $\dot{M}^B$  can be calculated by solving the following equation:

$$\hat{f}_c^B(\dot{M}^B) + \frac{1}{2}\dot{M}^B \star \hat{s}^s = \hat{f}_{c,0}^B, \quad (16)$$

where  $\hat{f}_{c,0}^B$  is equal to  $\hat{f}_c^B$  minus the term due to  $\dot{M}^B$ , i.e.,  $\hat{f}_{c,0}^B = \hat{f}_c^B - (-\frac{1}{2}\dot{M}^B \star \hat{s}^s)$ .

**Remark 4.** It can be easily shown that the model-dependent version of the control law given by Eq. (6), i.e.,

$$\begin{aligned} \hat{f}_c^B = & -M^B \star \hat{a}_g^B - \frac{3\mu\hat{r}_{B/I}^B}{\|\hat{r}_{B/I}^B\|^5} \times (M^B \star (\hat{r}_{B/I}^B)^s) - M^B \star \hat{a}_{J_2}^B - \hat{f}_d^B - \text{vec}(\hat{q}_{B/D}^*(\hat{q}_{B/D}^s - \hat{1}^s)) - K_d \star \hat{s}^s - \frac{1}{2}\dot{M}^B \star \hat{s}^s \\ & + \hat{\omega}_{B/I}^B \times (M^B \star (\hat{\omega}_{B/I}^B)^s) + M^B \star (\hat{q}_{B/D}^* \hat{\omega}_{D/I}^D \hat{q}_{B/D})^s + M^B \star (\hat{\omega}_{D/I}^B \times \hat{\omega}_{B/D}^B)^s - M^B \star (K_p \star \frac{d}{dt}(\hat{q}_{B/D}^*(\hat{q}_{B/D}^s - \hat{1}^s))), \end{aligned} \quad (17)$$

where the estimates of the dual inertia matrix and dual disturbance force are replaced by its true values, still guarantees that  $\lim_{t \rightarrow \infty} \hat{q}_{B/D} = \pm\hat{1}$  and  $\lim_{t \rightarrow \infty} \hat{\omega}_{B/D}^B = \hat{0}$ , for all initial conditions.

Sufficient conditions on the reference pose (i.e., on the reference position and attitude) can be derived that guarantee that the estimate of the dual inertia matrix will converge to the true dual inertia matrix. Since these conditions are the same conditions given in Proposition 1 of Ref. [7], with the additional assumptions that  $\dot{M}^B$  is known and  $v(M^B), v(\dot{M}^B), v(\ddot{M}^B) \in \mathcal{L}_\infty$ , the reader is referred to Ref. [7] for more details. Note however that the controller presented in Theorem 1 guarantees almost global asymptotical stability of the linear and angular position and velocity tracking errors with or without dual inertia matrix identification. Nevertheless, identification of the mass and inertia matrix of the satellite might be important, for example, for calculation of re-entry trajectories and terminal velocities, state estimation, fault-detecting-and-isolation systems, and docking/undocking scenarios.

## V. Simulation Results

Two numerical simulations are presented in this section. In the first, the proposed controller is applied to a conceivable satellite proximity operations scenario. In the second, the same controller is used to identify the time-varying mass and inertia matrix of the chaser satellite.

### A. Satellite Proximity Operations

In this numerical simulation, the versatility of the controller is demonstrated by using it, in sequence, to approach (phase #1), circumnavigate (phase #2), and dock (phase #3) with a target satellite.

Four reference frames are defined: the inertial frame, the target frame, the desired frame, and the body frame. The inertial frame is the Earth-Centered-Inertial (ECI) frame. The body frame is some frame fixed to the chaser satellite and centered at its center of mass. The target frame and the desired frame are defined as

$$\bar{I}_T = \frac{\bar{r}_{T/I}}{\|\bar{r}_{T/I}\|}, \quad \bar{J}_T = \bar{K}_T \times \bar{I}_T, \quad \bar{K}_T = \frac{\bar{\omega}_{T/I}}{\|\bar{\omega}_{T/I}\|}$$

and

$$\bar{I}_D = \frac{\bar{r}_{D/T}}{\|\bar{r}_{D/T}\|}, \quad \bar{J}_D = \bar{K}_D \times \bar{I}_D, \quad \bar{K}_D \parallel \bar{K}_T,$$

respectively, where  $\bar{\omega}_{T/I} = \frac{\bar{r}_{T/I} \times \dot{\bar{r}}_{T/I}}{\|\bar{r}_{T/I}\|^2}$  is calculated from the orbital angular momentum of the target spacecraft with respect to the inertial frame given by  $\bar{h}_{T/I} = m\|\bar{r}_{T/I}\|^2\bar{\omega}_{T/I} = \bar{r}_{T/I} \times m\dot{\bar{r}}_{T/I}$ . The target satellite is assumed to be fixed to the target frame. The objective of the control law is to superimpose the body frame to the desired frame. The relationship between the different frames is illustrated in Figure 1. Additional details

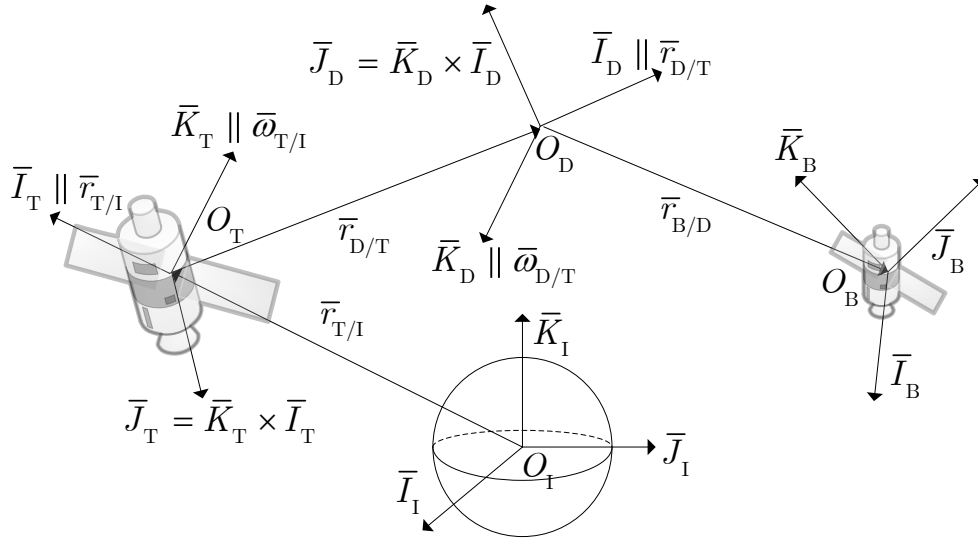


Figure 1. Reference frames.

about the simulation are given in Ref. [7].

Unlike in Ref. [7], the mass and inertia matrix of the chaser satellite are time-varying and given by

$$\bar{I}^B(t) = \frac{1}{2} \sin^2 \left( \frac{2\pi}{20} t \right) \bar{I}_0^B + \bar{I}_0^B \quad (18)$$

and

$$m(t) = 100 - 0.001t \text{ kg}, \quad (19)$$

where

$$\bar{I}_0^B = \begin{bmatrix} 22 & 0.2 & 0.5 \\ 0.2 & 20 & 0.4 \\ 0.5 & 0.4 & 23 \end{bmatrix} \text{ kg m}^2. \quad (20)$$

The form of Eq. (18) was taken from Ref. [12], whereas Eq. (19) corresponds to a linear decrease in mass with a flow rate orders of magnitude higher than the flow rate of electric thrusters, which is typically just a few milligrams per second.<sup>20</sup> As explained in Remark 3,  $\dot{M}^B$  and  $\dot{f}_c^B$  are typically coupled. However, for simplicity and without loss of generality, for the purpose of demonstrating the properties proven in Theorem 1,  $\dot{M}^B$  and  $\dot{f}_c^B$  are assumed to be decoupled here.

The constant disturbance force and torque acting on the chaser satellite are set to  $\bar{f}_d^B = [0.005, 0.005, 0.005]^T$  N and  $\bar{\tau}_d^B = [0.005, 0.005, 0.005]^T$  N m, respectively. The origin of the body frame (coincident with the center of mass of the chaser satellite) is positioned relative to the origin of the desired frame at  $\bar{r}_{B/D}^B = [2, 2, 2]^T$  m. The initial error quaternion, relative linear velocity, and relative angular velocity of the body frame with respect to the desired frame are set to  $q_{B/D} = [q_{B/D1} \ q_{B/D2} \ q_{B/D3} \ q_{B/D4}]^T = [0.4618, 0.1917, 0.7999, 0.3320]^T$ ,  $\bar{v}_{B/D}^B = [v_{B/D}^B \ w_{B/D}^B]^T = [0.1, 0.1, 0.1]^T$  m/s, and  $\bar{\omega}_{B/D}^B = [p_{B/D}^B \ q_{B/D}^B \ r_{B/D}^B]^T = [0.1, 0.1, 0.1]^T$  rad/s, respectively. The initial estimates for the mass, inertia matrix, and dual disturbance force are set to zero, whereas the control gains are chosen to be  $\bar{K}_r = 0.1I_3$ ,  $\bar{K}_q = 0.25I_3$ ,  $\bar{K}_v = 15I_3$ ,  $\bar{K}_w = 15I_3$ ,  $K_i = 100I_7$ ,  $\bar{K}_f = 0.8I_3$ , and  $\bar{K}_\tau = 0.8I_3$ .

Figure 2 shows the linear and angular velocity of the desired frame with respect to the inertial frame expressed in the desired frame for the complete maneuver. These signals form the reference for the controller.

Figure 3 shows the initial transient response and the transient response between phases #1 and #2 of the position and attitude of the body frame with respect to the desired frame using the controller given by Eq. (6) (adaptive) and the controller given by Eq. (17) (nonadaptive). Note that the transition between phases #1 and #2 occurs at 400 s. The transient response between phases #2 and #3 is similar and, thus, not shown here. Both controllers successfully cancel the relative position and attitude errors at the beginning of the maneuver and between phases, even with time-varying mass and inertia matrix. These latter are due to the fact that  $\bar{\omega}_{D/T}^T$  and  $\bar{v}_{D/T}^T$  are discontinuous between phases. In other words, between phases  $\bar{\omega}_{D/I}^D \notin \mathcal{L}_\infty$ , which instantaneously violates the conditions of Theorem 1.

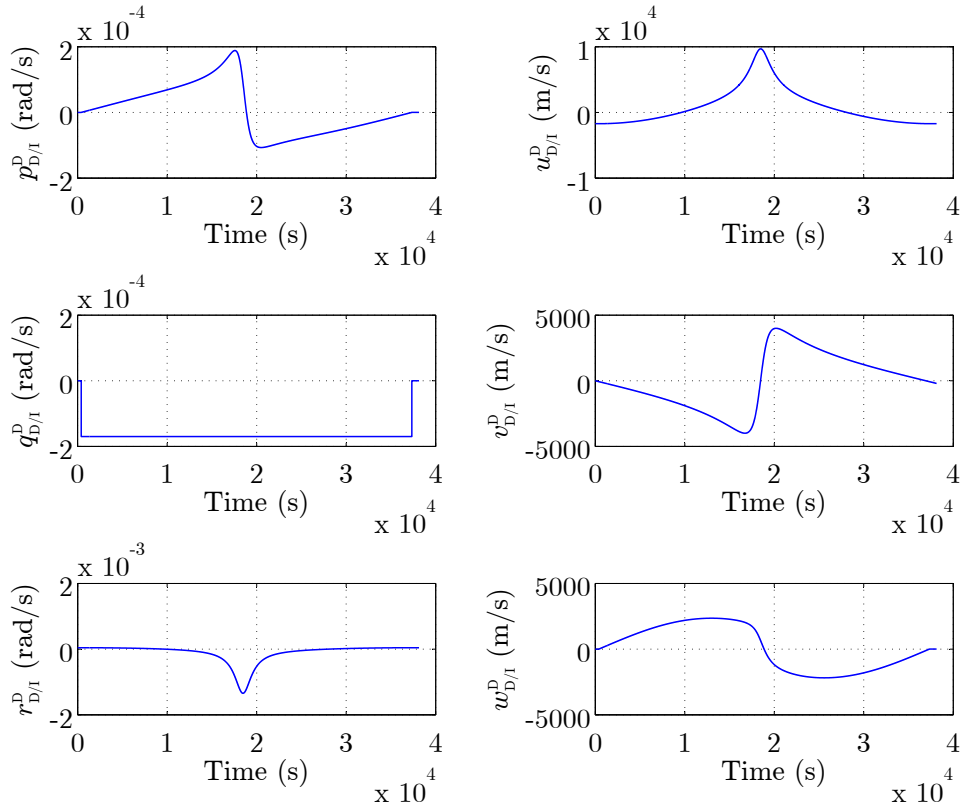


Figure 2. Reference motion.

Figure 4 shows the relative linear and angular velocity of the body frame with respect to the desired frame for the same two cases studied in Figure 3. Again, both controllers successfully cancel the relative linear and



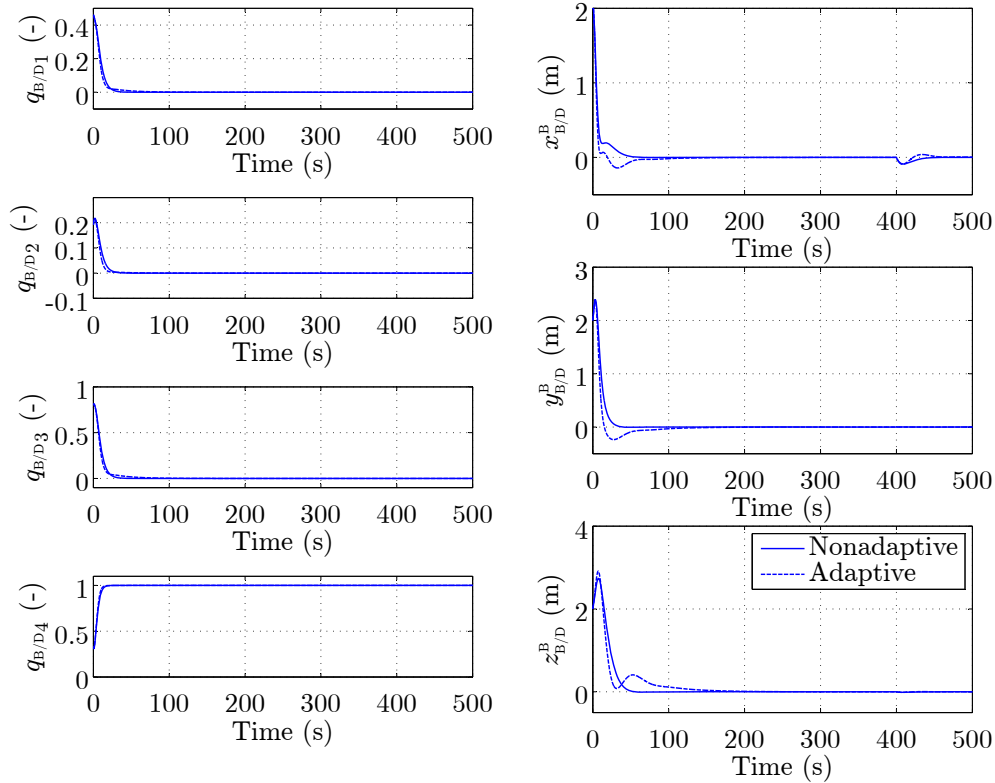


Figure 3. Relative attitude and position expressed in the body frame.

angular velocity errors at the beginning of the maneuver and between phases, even with time-varying mass and inertia matrix.

Figure 5 shows that the mass and inertia matrix estimates do not converge to their true values for this reference motion. This is to be expected as this reference motion is not designed to provide persistence of excitation. Nevertheless, the controller is still able to track the reference motion.

Figure 6 shows that for this reference motion, the dual disturbance force estimate converges to its true value. Note however that Theorem 1 only guarantees that this estimate will remain uniformly bounded. Relatively small oscillations in the estimate can be seen between phases as a result of the discontinuities in  $\hat{\omega}_{D/I}^D$ .

Finally, Figure 7 shows the control force,  $\bar{f}_c^B = [f_{c1}^B \ f_{c2}^B \ f_{c3}^B]^T$ , and the control torque,  $\bar{\tau}_c^B = [\tau_{c1}^B \ \tau_{c2}^B \ \tau_{c3}^B]^T$ , produced by the adaptive and nonadaptive controllers during the initial transient response and between phases #1 and #2.

## B. Mass and Inertia Matrix Identification

In this numerical simulation, the control law is used to identify the time-varying mass and inertia matrix of a satellite in a Geosynchronous Earth Orbit (GEO) with orbital elements given in Table 1 of Ref. [7].

In this scenario, the target frame is the unperturbed Hill frame<sup>21</sup> of the satellite. Note that in this case there is not a physical spacecraft attached to the target frame. The desired frame is defined to have the same pose as the target frame at the beginning of the simulation. The inertial frame and the body frame are defined as in the previous example.

The satellite has the same time-varying mass and inertia matrix as the chaser satellite in the previous example. As assumed in Proposition 1 of Ref. [7], the dual disturbance force is assumed to be known and, in this example, equal to zero. The body frame is assumed to have the same position, attitude, linear velocity, and angular velocity as the desired frame at the beginning of the simulation. The initial estimates for the mass and inertia matrix are set to zero. The control gains are the same as in the previous simulation.

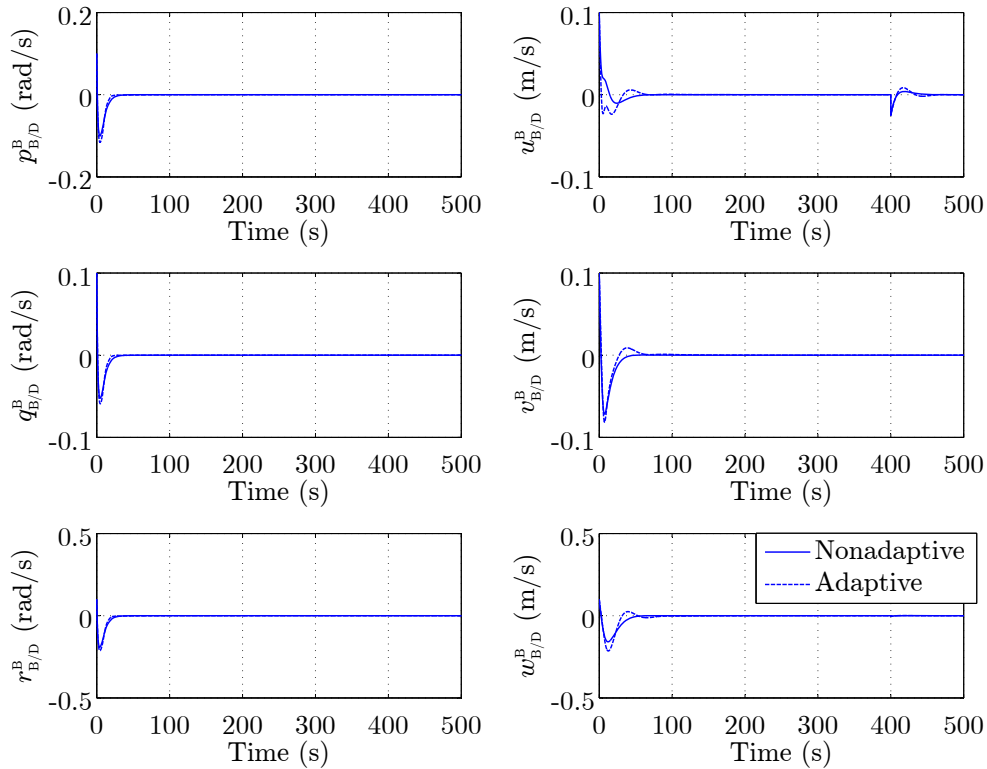


Figure 4. Relative linear and angular velocity expressed in the body frame.

The relative motion of the desired frame with respect to the target frame is defined in Figure 8. It is composed by a pure translation and several pure rotations designed to identify the mass and the elements of the inertia matrix in sequence, while keeping the control forces and torques within reasonable values. This reference motion is the same reference motion used in Ref. 7 to identify the constant mass and inertia matrix of the chaser satellite.

The identification of the time-varying mass and inertia matrix is shown in Figure 9. Note that the time-varying mass and inertia matrix are identified even though their initial estimates are zero. They are identified in sequence:  $m$  is identified during the first triangle waveform (on  $v_{D/T}^D$ ),  $I_{12}$ ,  $I_{22}$ , and  $I_{23}$  are identified during the second triangle waveform (on  $q_{D/T}^D$ ),  $I_{11}$  and  $I_{13}$  are identified during the third triangle waveform (on  $p_{D/T}^D$ ), and  $I_{33}$  is identified during the fourth and last triangle waveform (on  $r_{D/T}^D$ ). The associated control forces and torques are shown in Figure 10.

## VI. Conclusion

An adaptive pose tracking controller for satellites with unknown and time-varying mass and inertia matrix is presented. The controller is proved to ensure almost global asymptotic stability of the linear and angular position and velocity tracking errors. The gravity force, the gravity-gradient torque, Earth's oblateness, and unknown constant disturbance forces and torques are considered. Moreover, sufficient conditions of the reference motion are specified that guarantee identification of the time-varying mass and inertia matrix of the satellite. This controller is especially suited for satellites with relatively high and quick variations of mass and moment of inertia, such as highly maneuverable small satellites equipped with relatively powerful thrusters and control moment gyros.

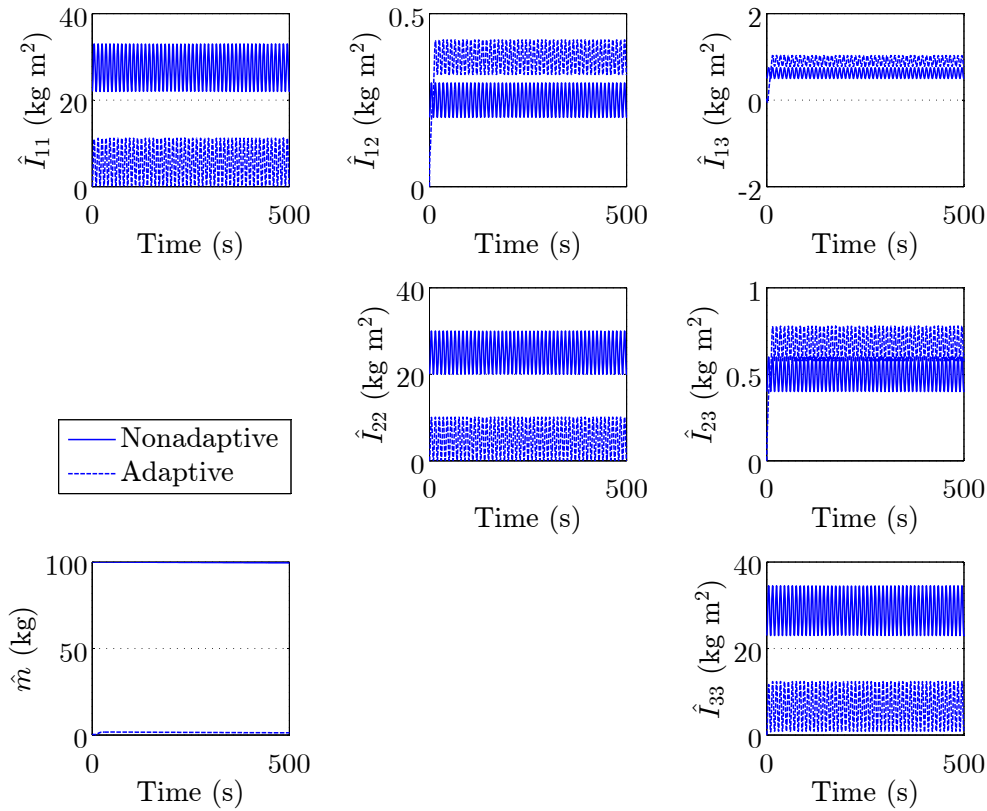


Figure 5. Mass and inertia matrix estimation for low-exciting reference motion.

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## References

- <sup>1</sup>Pan, H. and Kapila, V., “Adaptive Nonlinear Control for Spacecraft Formation Flying with Coupled Translational and Attitude Dynamics,” *Proceedings of the 40th IEEE Conference on Decision and Control*, Orlando, Florida USA, December 2001, pp. 2057–2062.
- <sup>2</sup>Singla, P., Subbarao, K., and Junkins, J. L., “Adaptive Output Feedback Control for Spacecraft Rendezvous and Docking Under Measurement Uncertainty,” *Journal of Guidance, Control, and Dynamics*, Vol. 29, No. 4, July-August 2006, pp. 892–902.
- <sup>3</sup>Tanygin, S., “Generalization of Adaptive Attitude Tracking,” *AIAA/AAS Astrodynamics Specialist Conference and Exhibit*, AIAA 2002-4833, Monterey, California, August 5-8 2002.
- <sup>4</sup>Wang, J. and Sun, Z., “6-DOF robust adaptive terminal sliding mode control for spacecraft formation flying,” *Acta Astronautica*, Vol. 73, April-May 2012, pp. 676–87.
- <sup>5</sup>Filipe, N. and Tsiotras, P., “Adaptive Model-Independent Tracking of Rigid Body Position and Attitude Motion with Mass and Inertia Matrix Identification using Dual Quaternions,” *AIAA Guidance, Navigation, and Control Conference*, AIAA 2013-5173, Boston, MA, August 19-22 2013.
- <sup>6</sup>Bhat, S. P. and Bernstein, D. S., “A topological obstruction to continuous global stabilization of rotational motion and the unwinding phenomenon,” *Systems & Control Letters*, Vol. 39, No. 1, January 28 2000, pp. 63 – 70.
- <sup>7</sup>Filipe, N. and Tsiotras, P., “Adaptive Position and Attitude-Tracking Controller for Satellite Proximity Operations Using Dual Quaternions,” *Journal of Guidance, Control, and Dynamics*, 2014, <http://arc.aiaa.org/doi/abs/10.2514/1.G000054>.
- <sup>8</sup>Vallado, D. and McClain, W., *Fundamentals of Astrodynamics and Applications*, Space Technology Library, Springer, 3rd ed., 2007, pp. 707–711 and 990.
- <sup>9</sup>Filipe, N. and Tsiotras, P., “Simultaneous Position and Attitude Control Without Linear and Angular Velocity Feedback Using Dual Quaternions,” *American Control Conference*, IEEE, Piscataway, NJ, Washington, DC, June 17-19 2013, pp. 4815–4820.

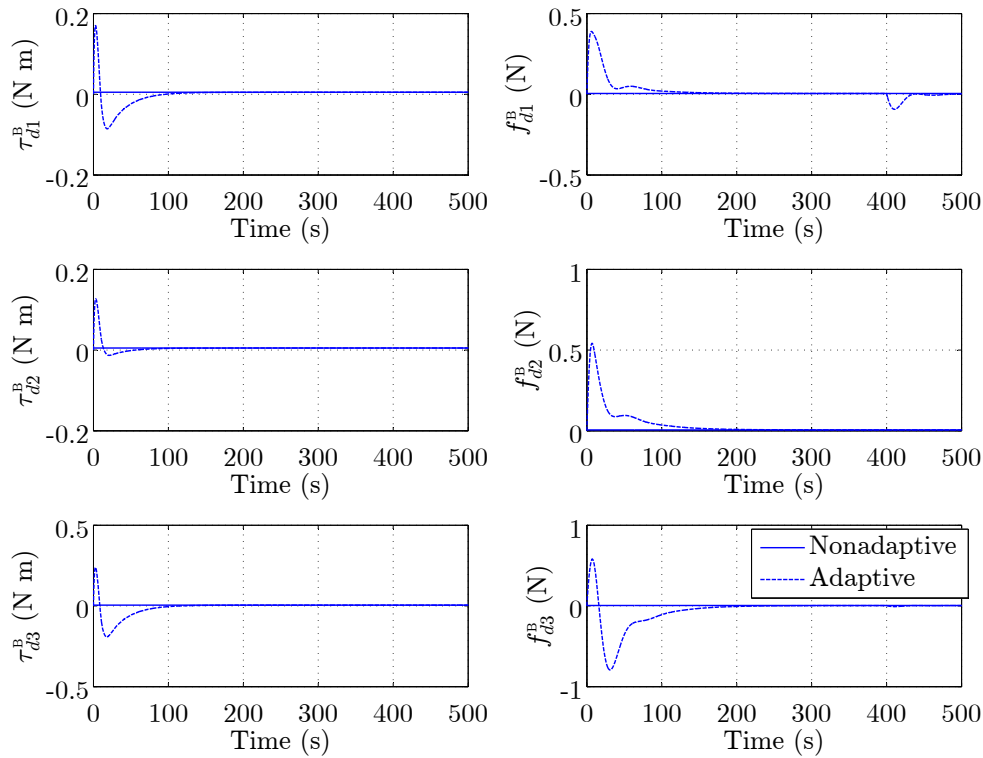


Figure 6. Dual disturbance force estimation.

<sup>10</sup>Filipe, N. and Tsiotras, P., “Rigid Body Motion Tracking Without Linear and Angular Velocity Feedback Using Dual Quaternions,” *European Control Conference*, IEEE, Piscataway, NJ, Zurich, Switzerland, July 17-19 2013, pp. 329–334.

<sup>11</sup>Yoon, H. and Tsiotras, P., “Spacecraft Adaptive Attitude and Power Tracking with Variable Speed Control Moment Gyroscopes,” *Journal of Guidance, Navigation, and Dynamics*, Vol. 25, No. 6, November-December 2002, pp. 1081–1090.

<sup>12</sup>Weiss, A., Kolmanovsky, I., and Bernstein, D. S., “Inertia-Free Attitude Control of Spacecraft with Unknown Time-Varying Mass Distribution,” *62nd International Astronautical Congress*, IAC-11-C1.5.9, Cape Town, South Africa, 2011.

<sup>13</sup>Han, D., Wei, Q., Li, Z., and Sun, W., “Control of Oriented Mechanical systems: A Method Based on Dual Quaternions,” *Proceeding of the 17th World Congress, The International Federation of Automatic Control*, International Federation of Automatic Control, Laxenburg, Austria, Seoul, Korea, July 6–11 2008, pp. 3836–3841.

<sup>14</sup>Wu, Y., Hu, X., Hu, D., Li, T., and Lian, J., “Strapdown Inertial Navigation System Algorithms Based on Dual Quaternions,” *IEEE Transactions on Aerospace and Electronic Systems*, Vol. 41, No. 1, January 2005, pp. 110–132.

<sup>15</sup>Wang, X. and Yu, C., “Unit dual quaternion-based feedback linearization tracking problem for attitude and position dynamics,” *Systems & Control Letters*, Vol. 62, No. 3, March 2013, pp. 225–233.

<sup>16</sup>Schaub, H. and Junkins, J., *Analytical Mechanics of Space Systems*, chap. 11.4 Spherical Harmonic Gravity Potential, Education Series, American Institute of Aeronautics & Astronautics, 2003.

<sup>17</sup>Haddad, W. and Chellaboina, V., *Nonlinear Dynamical Systems and Control: A Lyapunov-Based Approach*, Princeton University Press, 2008, p. 228.

<sup>18</sup>Han, D.-P., Wei, Q., and Li, Z.-X., “Kinematic Control of Free Rigid Bodies Using Dual Quaternions,” *International Journal of Automation and Computing*, Vol. 5, No. 3, July 2008, pp. 319–324.

<sup>19</sup>Mayhew, C. G., Sanfelice, R. G., and Teel, A. R., “Robust Global Asymptotic Attitude Stabilization of a Rigid Body by Quaternion-based Hybrid Feedback,” *Joint 48th IEEE Conference on Decision and Control and 28th Chinese Control Conference*, Shanghai, P.R. China, December 16-18 2009, pp. 2522–2527.

<sup>20</sup>Martinez-Sanchez, M. and Pollard, J. E., “Spacecraft Electric Propulsion - An Over,” *Journal of Propulsion and Power*, Vol. 14, No. 5, September-October 1998, pp. 688–699.

<sup>21</sup>Kristiansen, R., Nicklasson, P. J., and Gravdahl, J. T., “Spacecraft coordination control in 6DOF: Integrator backstepping vs passivity-based control,” *Automatica*, Vol. 44, No. 11, November 2008, pp. 2896–2901.

<sup>22</sup>Ahmed, J., Coppola, V. T., and Bernstein, D. S., “Adaptive Asymptotic Tracking of Spacecraft Attitude Motion with Inertia Matrix Identification,” *Journal of Guidance, Control, and Dynamics*, Vol. 21, No. 5, September-October 1998, pp. 684–691.

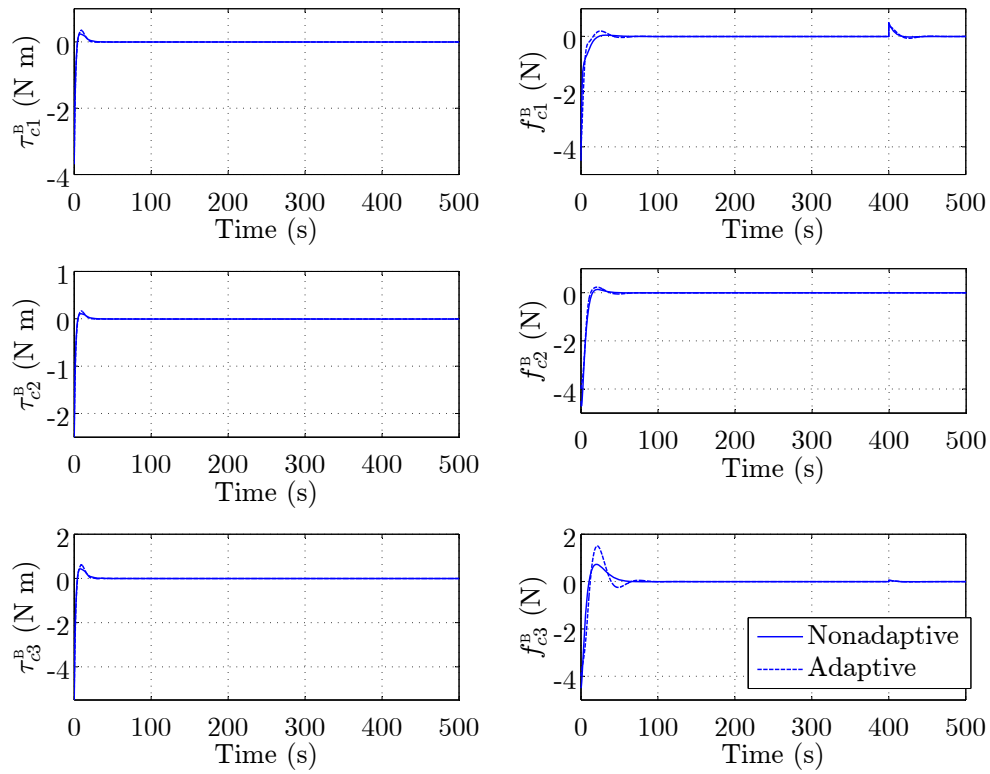


Figure 7. Control force and torque.

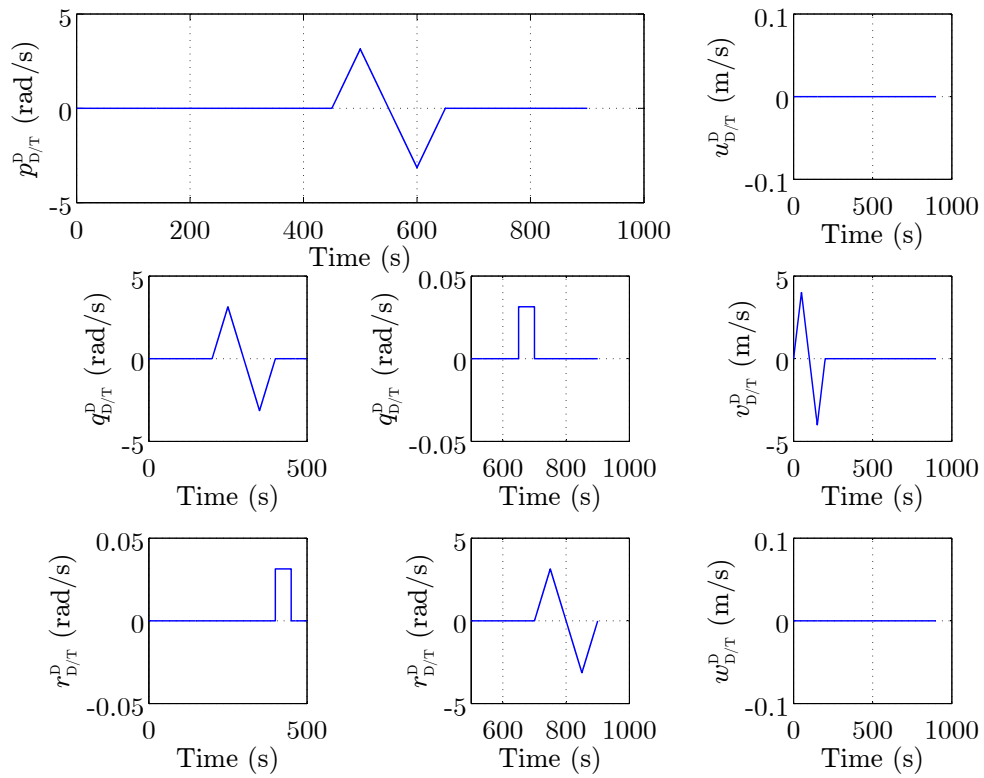


Figure 8. Reference motion for identification.

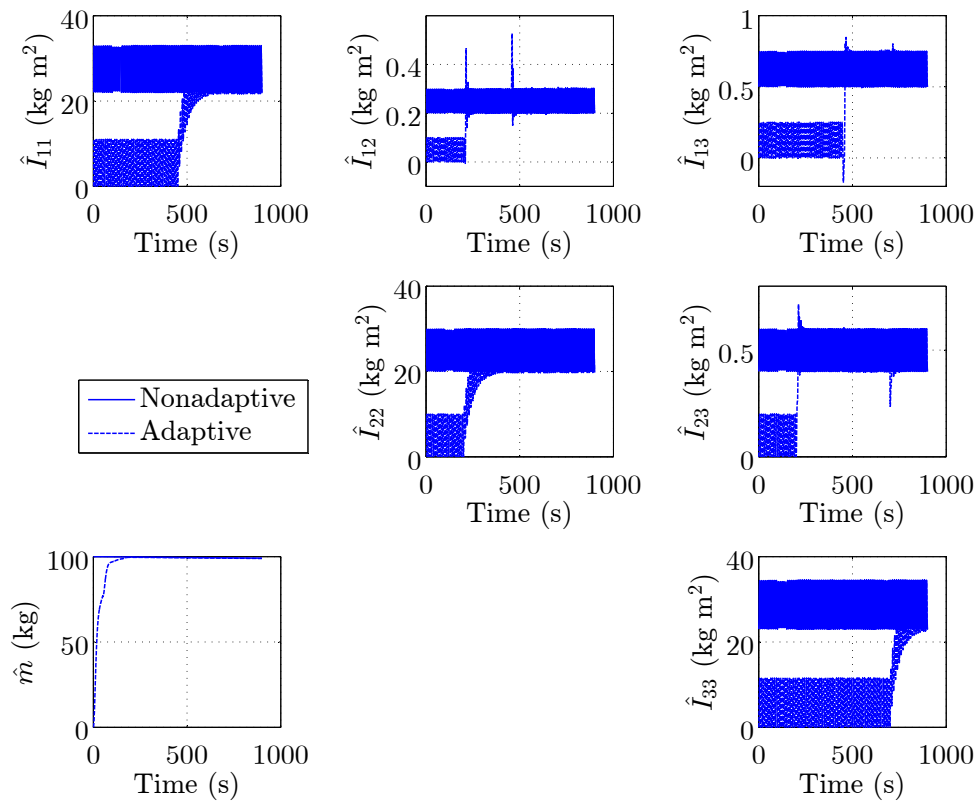


Figure 9. Mass and inertia matrix identification.

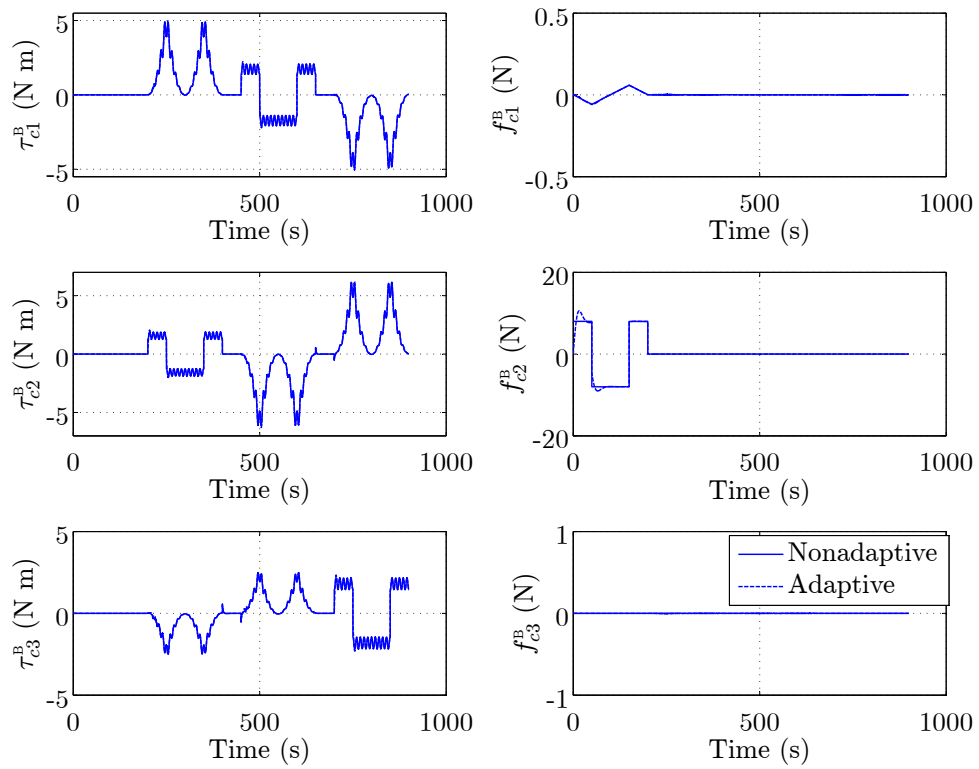


Figure 10. Control force and torque during identification.